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MATHEMATICS

On Graphs Whose Local Subgraphs Are Strongly Regular with Parameters (162, 21, 0, 3)

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We consider undirected graphs without loops or multiple edges. If a and b are vertices in a graph Γ, then $d(a, b)$ denotes the distance between a and b, and $\Gamma_i(a)$ denotes the subgraph of Γ induced by the set of vertices of Γ that are a distance of i away from a. The subgraph $\Gamma_1(a)$ is called the neighborhood of a and is denoted by $[a]$.

Γ is called a regular graph of degree k if $[a]$ contains precisely k vertices for any vertex a in Γ. Γ is called an edge-regular graph with parameters (v, k, λ) if Γ is a regular graph of degree k on v vertices and each edge in Γ lies in λ triangles. Γ is said to be an amply regular graph with parameters (v, k, λ, μ) if Γ is an edge-regular graph with the corresponding parameters and the subgraph $[a] \cap [b]$ contains μ vertices in the case of $d(a, b) = 2$. An amply regular graph of diameter 2 is called a strongly regular graph.

A graph Γ of diameter d is said to be antipodal if the relation of coincidence or being a distance of d apart on its vertex set is an equivalence relation. An antipodal quotient Γ' is a graph whose vertices are the antipodal classes of Γ and two classes are adjacent if a vertex of one class is adjacent to a vertex of the other class. An antipodal graph Γ is called an r-covering (of its antipodal quotient) if each of its antipodal classes contains precisely r vertices.

If vertices u and w are separated by a distance of i in Γ, then $b_i(u, w)$ ($c_i(u, w)$) denotes the number of vertices in the intersection of $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is called a distance-regular graph with the intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of the vertices u and w separated by a distance of i. Let $a_i = k - b_i - c_i$.

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Distance-regular graphs with strongly regular local subgraphs were characterized in [1]. An issue of special interest is the case where the neighborhoods of vertices do not contain triangles. The known strongly regular graph without triangles is a complete bipartite graph or is isomorphic to the Moore graph (graph with parameters $(k^2 + 1, k, 0, 1)$ for $k = 2, 3, 7$), to the complement graph of the Clebsch graph, to the Gewirtz graph, Higman–Sims graph, or Mathieu graph (to the second neighborhood of a vertex in the Higman–Sims graph). The known strongly regular Moore graph is a pentagon, the Petersen graph, or the Hoffman–Singleton graph.

Thus far, the distance-regular graphs whose local subgraphs are isomorphic to a known strongly regular graph without triangles have been classified (see [2–6]).

Proposition. *Let Γ be a distance-regular graph whose local subgraphs are isomorphic to a known strongly regular graph Γ without triangles. Then one of the following assertions holds:*

(1) $\Delta = K_{2\times n}$ *and* $\Gamma = K_{3\times n}$.

(2) Δ *is a pentagon, and* Γ *is an icosahedron graph.*

(3) Δ *is the Petersen graph, while* Γ *is the complement graph of the triangular graph* $T(7)$ *or a graph with the intersection array* $\{10, 6, 4, 1; 1, 2, 6, 10\}$ (*Conway–Smith graph*) *or* $\{10, 6, 4; 1, 2, 5\}$ (*Doro graph*).

(4) Δ *is a strongly regular graph with parameters* $(16, 5, 0, 2)$, *and* Γ *is a graph with the intersection array* $\{16, 10, 1; 1, 5, 16\}$.

(5) Δ *is the Hoffman–Singleton graph, and* Γ *is the Terwilliger graph with the intersection array* $\{50, 42, 1; 1, 2, 50\}$ *or* $\{50, 42, 9; 1, 2, 42\}$.

(6) Δ *is the Gewirtz graph and either*

(i) Γ *is a strongly regular graph with parameters* $(162, 56, 10, 24)$ *or* $(372, 56, 10, 8)$ *or*

(ii) Γ *has the intersection array* $\{56, 45, 36, 1; 1, 8, 45, 56\}$ (*Soicher graph*).

(7) Δ *is the Higman–Sims graph, and* Γ *is a strongly regular graph with parameters* $(486, 100, 22, 20)$.

In this paper, we classify the distance-regular graphs in which the neighborhood of vertices are isomorphic to a strongly regular graph with parameters $(162, 21, 0, 3)$.

Theorem. *Let Γ be a distance-regular graph whose local subgraphs are strongly regular with parameters* (162, 21, 0, 3). *Then Γ is a strongly regular graph with parameters* (289, 162, 21, 36) *or* (2431, 162, 21, 10).

Below are some auxiliary results.

Lemma 1 [7, Lemma 3.1]**.** *Let Γ be a strongly regular graph with parameters (v, k, λ, μ). Then either $k = 2\mu$ and $\lambda = \mu - 1$ (so-called half case) or the nonprincipal eigenvalues $n - m$ and $-m$ of Γ are integers, where $n^2 = (\lambda - \mu)^2 + 4(k - \mu)$, $n - \lambda + \mu = 2m$, and the multiplicity of $n - m$ is $\frac{k(m-1)(k+m)}{\mu n}$. Furthermore, if m is an integer greater than 1, then $m - 1$ divides $k - \lambda - 1$ and*

$$\mu = \lambda + 2 + (m-1) - \frac{k-\lambda-1}{m-1},$$

$$n = m - 1 + \frac{k-\lambda-1}{m-1}.$$

Lemma 2. *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) and Δ be an induced subgraph on N vertices with M edges and vertex degrees $d_1, d_2, \ldots, d_N$. Then*

$$(v-N) - (kN - 2M) + \left(\lambda M + \mu\left(\binom{N}{2} - M\right) - \sum_{i=1}^{N}\binom{d_i}{2}\right) = x_0 + \sum_{i=3}^{N}\binom{i-1}{2}x_i,$$

where $x_i = x_i(\Delta)$.

Proof. Calculating the number of vertices in $\Gamma - \Delta$, the number of edges between Δ and $\Gamma - \Delta$, and the number of triplets of the form $(a, \{b, c\})$, where $a \in \Gamma - \Delta$, b, and $c \in \Delta \cap [a]$, we obtain the relations

$$v - N = \sum x_i, \quad kN - 2M = \sum i x_i,$$

$$\lambda M + \mu\left(\binom{N}{2} - M\right) - \sum_{i=1}^{N}\binom{d_i}{2} = \sum\binom{i}{2}x_i.$$

Subtracting the second equation from the sum of the first and third ones yields the required result.

Lemma 3. *Suppose that Γ is a strongly regular graph with parameters* (162, 21, 0, 3), *Δ is a regular subgraph of Γ of degree 3 on n vertices, X_i is the set of vertices from $\Gamma - \Delta$ that are adjacent to precisely i vertices in Δ, and $x_i = |X_i|$. Then the following assertions hold:*

(1) $\sum x_i = 162 - n$, $\sum i x_i = 18n$, *and*

$$\sum\binom{i}{2}x_i = 3(n^2 - 10n)/2.$$

$$x_0 + \sum\binom{i-1}{2}x_i = 162 + \frac{3n^2 - 56n}{2}.$$

(2) *The parameter n is not equal to* 6.

(3) *If $n = 8$, then $x_0 \le 34$; if $n = 10$, then $x_0 \le 32$; and if $n = 12$, then $x_0 \le 42$.*

(4) $n \le 69$.

Proof. By Lemma 2, we have $\sum x_i = 162 - n$, $\sum i x_i = 18n$, $\sum\binom{i}{2}x_i = 3\left(\binom{n}{2} - \frac{3n}{2}\right) - 3n = \frac{3n^2 - 18n}{2}$. Therefore, $x_0 + \sum\binom{i-1}{2}x_i = 162 + \frac{3n^2 - 56n}{2}$. Assertion (1) is proved.

If $n = 6$, then $x_i = 0$ for $i \ge 2$. Therefore, $x_0 = 48$, $x_1 = 108$, and each vertex from X_0 is adjacent to 18 vertices from X_1. From this, the number of edges between X_0 and X_1 is $48 \cdot 18$. A vertex from X_1 that is adjacent to a vertex $a \in \Delta$ has, in X_1, two neighbors adjacent to vertices from $\Delta(a)$, three neighbors adjacent to vertices from $\Delta - a^{\perp}$, and nine neighbors in X_0. Therefore, the number of edges between X_0 and X_1 is $108 \cdot 9$, a contradiction. Thus, Γ does not contain $K_{3,3}$-subgraphs. Assertion (2) is proved.

Let $n = 8$. Then $x_0 \le 162 - \frac{8^2(7-3)}{2} = 34$. If $n = 10$, then $x_0 \le 162 - 5(56 - 30) = 32$. If $n = 12$, then $x_0 \le 162 - 6(56 - 36) = 42$. Assertion (3) is proved.

We have $-6 \le 3 - \frac{18n}{162 - n} \le 3$. Therefore, $7n \le 486$ and $n \le 69$.

Lemma 4. *Let Γ be a strongly regular graph with parameters* (162, 21, 0, 3)*, and let X and Y be subsets of vertices of Γ such that there are no edges between X and Y. Then $|X| \cdot |Y| \le \frac{(162 - |X|)(162 - |Y|)}{5^2}$; moreover, if $|X| = |Y|$, then $|X| \le 27$.*

Proof. Since there are no edges between X and Y, Proposition 4.6.1 in [8] implies that $|X| \cdot |Y| \le \frac{(v - |X|)(v - |Y|)(\theta_2 - \theta_1)^2}{(2k - \theta_2 - \theta_1)^2}$, where $\theta_1 = 3$ and $\theta_2 = -6$ are nonprincipal eigenvalues of Γ. Therefore, $|X| \cdot |Y| \le \frac{(162 - |X|)(162 - |Y|) \cdot 9^2}{45^2}$.

If $|X| = |Y|$, we have $5|X| \le 162 - |X|$ and $|X| \le 27$.

Lemma 5. *Let Γ be a distance-regular graph of diameter $d \ge 3$ whose local subgraphs are strongly regular with parameters* (162, 21, 0, 3), *and let $\theta_0 = k > \theta_1 > \ldots > \theta_d$ be the eigenvalues of Γ. Then $3 < \theta_1 \le 27$ and $-6 > \theta_d \ge -36$.*

Proof. By the Terwilliger theorem [9, Theorem 4.4.3], it holds that $-6 \geq b^- = -1 - \dfrac{b_1}{\theta_1 + 1}$, $3 \leq b^+ = -1 - \dfrac{b_1}{\theta_d + 1}$. Therefore, $\theta_1 \leq 27$ and $\theta_d \geq -36$. By [9, Corollary 3.7], we have $\theta_d < -6 < 3 < \theta_1$.

Lemma 6. *Let* Γ *be an amply regular graph of diameter d whose local subgraphs are strongly regular with parameters* (162, 21, 0, 3). *Then the following assertions hold:*

(1) *If the diameter of* Γ *is* 2, *then* Γ *has parameters* (289, 162, 21, 36) *or* (2431, 162, 21, 10).

(2) *If the diameter of* Γ *is larger than* 2, *then* $\mu \in \{8, 10, 12, 14, 18, 20, 24, 28, 30, 36, 40, 42, 54, 56, 60\}$.

(3) *If the diameter of* Γ *is larger than* 3, *then* $\mu \in \{8, 10, 12, 14, 18, 20, 24\}$.

Proof. By assumption, $k = 162$ and $\lambda = 21$. If the diameter of Γ is 2, then, by Lemma 1, the number $(\lambda - \mu)^2 + 4(k - \mu)$ is the square of a positive integer n. Therefore, $(\mu - 23)^2 + 560 = n^2$ and $(\mu, n) \in \{(54, 39), (36, 27), (10, 27)\}$. Therefore, Γ has eigenvalues 3, −36; 6; −21; or 19, −8. In the first case, the multiplicities of the eigenvalues are not integer. Therefore, Γ has parameters (289, 162, 21, 36) or (2431, 162, 21, 10). Assertion (1) is proved.

Let the diameter of Γ be larger than 2. By Lemma 3, we have $\mu \leq 68$ and $\mu \neq 6$. Since μ is an even divisor of $162 \cdot 140$, we have $\mu \in \{8, 10, 12, 14, 18, 20, 24, 28, 30, 36, 40, 42, 54, 56, 60\}$. Assertion (2) is proved.

Let the diameter of Γ be larger than 3, and let u, w, x, y, z be a geodesic 4-path in Γ. Then, in the graph $[x]$, there are no edges between $[u] \cap [x]$ and $[x] \cap [z]$ and, by Lemma 4, we have $\mu \leq 26$. Therefore, $\mu \in \{8, 10, 12, 14, 18, 20, 24\}$.

Remark. Let Δ be a strongly regular graph with parameters (289, 162, 21, 36) or (2431, 162, 21, 10) and Γ be a distance-regular graph of diameter d that is an r-covering of Δ. Then $\mu_\Gamma \geq 6$. If $d = 6$, then Γ has the intersection array $\{162, 140, t(r - 1), 36, 1; 1, 36, t, 140, 162\}$ or $\{162, 140, t(r - 1), 10, 1; 1, 10, t, 140, 162\}$. In this case, there are no admissible arrays.

If $d = 4$, then Γ has the intersection array $\left\{162, 140, \dfrac{36(r-1)}{r}; 1; 1, \dfrac{36}{r}, 140, 162\right\}$, r divides 6, the new eigenvalues θ_1 and θ_3 of Γ are the roots of the quadratic equation $x^2 - \lambda x - k = 0$, and the multiplicity of θ_1 is $m_1 = \dfrac{(r-1)v}{2 + \lambda\theta_1/k}$. Therefore, $\theta_1 = 27$ and $\theta_3 = -6$. In this case, there are no admissible arrays.

Throughout the rest of this paper, we assume that Γ is a distance-regular graph of diameter $d \geq 3$ whose local subgraphs are strongly regular with parameters (162, 21, 0, 3). Let u be a vertex of Γ and $k_i = |\Gamma_i(u)|$.

Lemma 7. *The parameter* μ *is at most* 24.

Proof. Let $\mu > 24$. By Lemma 6, the diameter of Γ is 3.

If $\mu \geq 54$, then, by Lemma 4, we have $b_2 \leq 12$. Computer calculations show that $\theta_3 < -36$, a contradiction to Lemma 5.

Let $\mu = 42$. Then $k_2 = 540$. By Lemma 4, we have $b_2 \leq 16$. If $c_3 \leq 132$, then $\theta_1 > 27$, a contradiction. Therefore, $c_3 \in \{135, 140, 144, 150, 156, 160, 162\}$. Only if $c_3 = 135$ and $b_2 = 2$ or $b_2 = 14$, all the eigenvalues are integers: 27, 4, −27 and 27, 1, −36. In any case, there are no admissible intersection arrays.

The other cases $\mu = 30, 36, 40$ are treated in a similar manner.

Lemma 8. *The diameter of* Γ *is larger than* 3.

Proof. Assume that the diameter of Γ is 3.

Let $\mu = 24$. Then $k_2 = 27 \cdot 35$ and b_2 is even. By Lemma 4, we have $b_2 \leq 30$. Furthermore, b_2 is divided by 6 and, if $c_3 < 135$, then $\theta_1 > 27$, a contradiction. From this, $c_3 \in \{135, 140, 150, 162\}$. In any case, there are no admissible intersection arrays. Let $\mu = 20$. Then $k_2 = 81 \cdot 14$. By Lemma 4, we have $b_2 \leq 35$. If $c_3 < 135$, then $\theta_1 > 27$, a contradiction. Therefore, $c_3 \in \{135, 136, 138, 140, 144, 150, 154, 156, 162\}$. In any case, there are no admissible intersection arrays.

Let $\mu = 18$. Then $k_2 = 9 \cdot 140$. By Lemma 4, we have $b_2 \leq 39$. Furthermore, b_2 is divided by 9 and, if $c_3 < 135$, then $\theta_1 > 27$, a contradiction. From this, $c_3 \in \{135, 140, 144, 162\}$. In any case, there are no admissible intersection arrays.

The other cases $\mu = 8, 10, 12, 14$ are treated in a similar fashion.

Lemma 9. *It is true that* $\mu \leq 12$.

Proof. Suppose that $d \geq 4$.

Let $\mu = 24$. Then $k_2 = 27 \cdot 35$, b_2 is divided by 6 and, by Lemma 4, we have $24 \leq b_2 \leq 30$. If $(c_3, c_4) \neq (140, 162)$, then $\theta_1 > 27$, a contradiction. Therefore, $(c_3, c_4) = (140, 162)$. Only if $b_3 = 1$ and $b_2 = 30$, Γ has integer eigenvalues: 27, 3, −6, −26. In any case, there are no admissible intersection arrays.

The other cases $\mu = 14, 18, 20$ are treated in a similar manner.

Lemma 10. *The diameter of* Γ *is larger than* 4.

Proof. Suppose that $d = 4$.

Let $\mu = 12$. Then $k_2 = 27 \cdot 140$. By Lemma 3, $12 \leq b_2 \leq 42$ and b_2 is divided by 3. If $(c_3, c_4) \neq (140, 162)$, then $\theta_1 > 27$, a contradiction. Therefore, $(c_3, c_4) = (140, 162)$. In the cases $b_3 = 1$, $b_2 = 15$, $b_2 = 24$, and $b_2 = 42$, Γ has integer eigenvalues: 27, 9, −6, −15; 27, 6, −6, −21; and 27, 3, −6, −36. In any case, there are no admissible intersection arrays.

The other cases $\mu = 8, 10$ are treated in a similar manner.

Lemma 11. *The following assertions hold:*

(1) *If* $\mu = 10, 18, 20$, *then* $d \leq 5$.

(2) *If* $\mu = 8, 12, 14$, *then* $d \leq 6$.

Proof. We have $c_3 - b_3 \geq c_2 - b_2 + 23$, ..., $c_i - b_i \geq c_{i-1} - b_{i-1} + 23$. Summing up the inequalities termwise produces $c_i - b_i \geq c_2 - b_2 + (i-2) \cdot 23$.

If $\mu \geq 18$, then, by Lemma 4, we have $b_2 \leq 39$ and $c_3 - b_3 \geq 18 - b_2 + 23$. Therefore, $d \leq 5$.

If $\mu = 14$, then, by Lemma 4, we have $b_2 \leq 48$, $c_4 - b_4 \geq 14 - b_2 + 46$, $c_4 \geq b_4 + 12$, and $d \leq 7$. If $d = 7$, we obtain $c_3 - b_3 \geq 14 - 48 + 23$ and $c_4 \leq b_3 \leq c_3 + 11 \leq b_4 + 11$, a contradiction.

If $\mu = 12$, then $k_2 = 27 \cdot 70$ and, by Lemma 3, we have $b_2 \leq 42$, $c_4 - b_4 \geq 12 - 42 + 46$, $c_4 \geq b_4 + 16$, and $d \leq 7$. If $d = 7$, we obtain $c_3 - b_3 \geq 12 - b_2 + 23$ and $c_4 \leq b_3 \leq c_3 + 7 \leq b_4 + 7$, a contradiction.

If $\mu = 10$, then $k_2 = 81 \cdot 28$ and, by Lemma 3, we have $b_2 \leq 32$. Therefore, $c_3 - b_3 > 10 - b_2 + 23$ and $d \leq 5$.

If $\mu = 8$, then $k_2 = 60 \cdot 23$ and, by Lemma 3, we have $b_2 \leq 34$. Therefore, $c_4 - b_4 \geq 8 - b_2 + 34$, $c_4 \geq b_4 + 8$, and $d \leq 7$. If $d = 7$, we obtain $c_3 - b_3 \geq 8 - 34 + 23$ and $c_4 \leq b_3 \leq c_3 + 3 \leq b_4 + 3$, a contradiction.

Lemma 12. *If* $d = 5$, *then* $\theta_1 > 76$.

Proof. Let $\mu = 20$. Then $k_2 = 81 \cdot 14$ and, by Lemma 4, we have $c_3 \leq b_2 \leq 35$. In any case, $\theta_1 > 99$.

Let $\mu = 18$. Then $k_2 = 9 \cdot 140$ and, by Lemma 4, we have $c_3 \leq b_2 \leq 39$. In any case, $\theta_1 > 92$.

Let $\mu = 14$. Then $k_2 = 81 \cdot 20$ and, by Lemma 4, we have $c_3 \leq b_2 \leq 48$. In any case, $\theta_1 > 76$.

The other cases $\mu = 8, 10, 12$ are considered in a similar fashion.

Lemma 13. *If* $d = 6$, *then* $\theta_1 > 49$.

Proof. Let $\mu = 14$. Then $k_2 = 81 \cdot 20$ and, by Lemma 4, we have $c_3 \leq b_2 \leq 48$. In any case, $\theta_1 > 55$.

The other cases $\mu = 8, 12$ are treated in a similar manner. The lemma, together with the theorem, is proved.

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